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CREEPING FLOW OF A POWER LAW FLUID PAST A FLUID SPHERE

V. MOHAN† and D. VENKATESWARLU
Department of Chemical Engineering, Indian Institute of Technology, Madras 600036, India

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Abstract—The stress variational principle is employed to obtain the lower bound for the drag offered by the creeping flow of a power law fluid past a Newtonian fluid sphere. In spite of the unprescribed interfacial velocity, a bound-on-bound approach yields bounds that are close to the upper bound obtained by Mohan (1974). Furthermore, for very viscous drops (solid behavior) the theory gives lower bounds that differ considerably from those of Wasserman & Slattery (1964) and show good agreement with the results of Yoshioka & Adachi (1973). The approach presented in this work provides an insight into the method of analyzing multiphase flow situations involving non-Newtonian fluids.

INTRODUCTION

Variational principles for the creeping flow of a non-Newtonian fluid were developed by Johnson (1960) using calculus of variations and by Slattery (1972) using a function space approach. Similar principles were developed by Keller, Rubenfeld & Molyneux (1967), for the flow of a suspension of solid and fluid particles in a fluid medium. This analysis is limited to Newtonian flow situations.

Wasserman & Slattery (1964) used the function space approach and obtained upper and lower bounds for the drag offered to a solid sphere by a power law fluid. However, for flow behavior indices close to unity, the lower bound is larger than the upper bound, and differs from that given by Yoshioka & Adachi (1973). Nakano & Tien (1968) analyzed the flow of a power law fluid past a Newtonian fluid sphere using a combination of Galerkin and variational methods. The functional used by them to obtain the upper bound is in error (Finlayson 1972). The analysis for the upper bound was improved by Mohan (1974) by including the contribution to the work function from the internal fluid. It was shown by Nakano & Tien that a lower bound based on the stress variational principle could not be obtained because of the unprescribed velocity at the fluid–fluid interface. For this same reason, the variational principles developed by Finlayson (1972) for non-Newtonian fluids can not be used to obtain a bound on the energy dissipation rate. The present work is concerned with demonstrating that a bound on bound can be obtained using the function space approach. The lower bound so obtained is close to the available upper bound. The technique is useful for studying multiphase flow problems in liquid–liquid extraction, phase separation and allied processes. A knowledge of the hydrodynamics can then be used to analyze the heat/mass transfer problem.

STATEMENT OF THE PROBLEM

Consider the steady state, axisymmetric, creeping flow of a power law fluid past a Newtonian fluid sphere. The fluids are assumed to be free from surfactant impurities. The physical properties of the fluids do not vary. The constitutive equations for the internal and external fluids are given by

$$\boldsymbol{\tau} = 2\eta_i \mathbf{D} \qquad \text{(internal)}, \qquad [1]$$

$$\boldsymbol{\tau} = 2K(2D_i^j D_j^i)^{(n-1)/2}\mathbf{D} \qquad \text{(external)}, \qquad [2]$$

where $\boldsymbol{\tau}$ is the extra stress tensor, $\mathbf{D}$ the rate-of-deformation, η_i the viscosity of the Newtonian

†Present address: Department of Chemical Engineering, Illinois Institute of Technology, Chicago, IL 60616, U.S.A.

internal fluid, and K and n are the consistency index and flow behavior index of the power law external fluid. When the external fluid is also Newtonian, the flow behavior index n equals unity in [2] and in all subsequent equations. Both the fluids are assumed incompressible.

DEVELOPMENT OF THE STRESS VARIATIONAL PRINCIPLE

The equations of continuity and motion can be written in tensorial form as

$$v^i_{,i} = 0\,, \qquad [3]$$

$$\tau^{ij}_{,j} - p^{,i} + \rho f^i = 0, \qquad [4]$$

where v^i, τ^{ij} and f^i are components of the velocity vector, extra stress tensor and body force, p is the pressure and ρ the density.

Let us define the work function $\mathbf{E}$, the stress variational functional H_τ and the complementary work function E_c for the flow field (internal and external) as

$$\mathbf{E} = \int_0^{\mathrm{II}} \eta(\mathrm{II})\, \mathrm{dII}\,, \qquad [5]$$

$$H_\tau = -\int_{V(s)} E_c^* \,\mathrm{d}V + \int_{S(s)} (\tau^*_{ij} n_j - (p+\rho\phi)^* n_i) v_i \,\mathrm{d}S\,, \qquad [6]$$

and

$$E_c = \int_0^{\mathrm{II}_\tau} \frac{\mathrm{dII}_\tau}{4\eta(\mathrm{II}_\tau)}\,. \qquad [7]$$

The quantities II and II_τ are the second invariants of the rate-of-deformation and extra stress tensor, $\mathbf{n}$ the normal, and $V(s)$ and $S(s)$ represent the volume domain and bounding surface of the volume domain respectively. The velocity v_i appearing on the R.H.S. of [6] is the actual velocity on the bounding surface. This velocity is unknown at the interface. The asterisk(*) denotes values obtained from a trial extra stress tensor satisfying the equation of motion [4].

For H_τ evaluated on the basis of any trial stress function satisfying the equation of motion, it has been shown (Slattery 1972) that

$$\int_{V(s)} \mathbf{E}\,\mathrm{d}V \geq H_\tau\,. \qquad [8]$$

From the homogeneous nature of the function $\mathbf{E}$, we can write that

$$tr(\boldsymbol{\tau}\cdot\mathbf{D}) = (n+1)\mathbf{E}\,, \qquad [9]$$

which implies that the function $\mathbf{E}$ is proportional to the rate of energy dissipation per unit volume, the proportionality constant depending on the non-Newtonianism of the fluid.

The quantity of interest is the total energy dissipation rate given by

$$V_\infty F_d = \int_{V_i+V_0} tr(\boldsymbol{\tau}\cdot\mathbf{D})\,\mathrm{d}V\,, \qquad [10]$$

where V_∞ and F_d are the free stream velocity and drag force respectively. Combining [8] to [10], we get

$$V_\infty F_d \geq 2H_{\tau i} + (n+1)H_{\tau 0}\,. \qquad [11]$$

This establishes the bound on the rate of energy dissipation. In the evaluation of $H_{\tau i}$ using [6] the bounding surface $S(s)$ is the interface, on which the velocity is unspecified. For the evaluation of $H_{\tau 0}$, the bounding surfaces are a sphere of radius infinity on which the velocity is specified as the free stream velocity, and the interface where the velocity is unspecified. In the evaluation of the R.H.S. of inequality [11], the surface integrals over the interface for internal and external fluids do not cancel if $n \neq 1$ and pose a problem. However for a pseudoplastic fluid ($n \leq 1$), and inequality [11] becomes

$$V_\infty F_d \geq 2H_{\tau i} + (n+1)H_{\tau 0} \geq (n+1)(H_{\tau i} + H_{\tau 0}) = H. \qquad [12]$$

Evaluation of H poses no problem as the surface integrals at the interface now cancel. We have, therefore,

$$H = (n+1)\left[-\int_{V_i+V_0} E_c^* \,\mathrm{d}V + \int_{S(r=\infty)} (\tau_{ij}^* n_j - (p+\rho\phi)^* n_i) v_i \,\mathrm{d}S\right] \qquad [13]$$

where $S(r=\infty)$ denotes a sphere of radius infinity. Inequality [12] and equation [13] constitute the bound-on-bound principle.

EVALUATION OF THE BOUND-ON-BOUND

Employing the spherical polar co-ordinates, the dimensionless radial and tangential velocities are given in terms of the dimensionless stream function ψ by the relations,

$$v_r = \frac{V_r}{V_\infty} = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad [14]$$

$$v_\theta = \frac{V_\theta}{V_\infty} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \qquad [15]$$

where $r = R/a$, R the radius vector and a the drop radius.

With this definition, the equation of motion for the internal fluid becomes

$$D^4\psi_i = 0, \qquad [16]$$

where

$$D^4 = \left\{\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right\}^2. \qquad [17]$$

Assuming a trial stream function for the inside fluid as a set of complete functions (Nakano & Tien 1968), we have

$$\psi_i^* = (C_1 r^2 + C_2 r^3 + C_3 r^4)(1 - z^2), \qquad [18]$$

where $z = \cos\theta$.

This form satisfies the condition that as $r \to 0$, $(v_r)_i^*$ and $(v_\theta)_i^*$ remain finite. The boundary condition is given by

$$(v_r)_i = 0 \quad \text{at} \quad r = 1, \qquad [19]$$

which yields

$$C_1 + C_2 + C_3 = 0. \qquad [20]$$

Substituting the stream function in [16], it can be shown that

$$C_2 = 0. \tag{21}$$

Using [20] and [21], the components of the extra stress function for the internal fluid become

$$\begin{aligned}
(\tau_{rr})_1^* &= 8\eta_i \frac{V_\infty}{a} C_1 z r, \\
(\tau_{r\theta})_1^* &= -6\eta_i \frac{V_\infty}{a} C_1 (1-z^2)^{1/2} r, \\
(\tau_{\theta\theta})_1^* = (\tau_{\phi\phi})_1^* &= -4\eta_i \frac{V_\infty}{a} C_1 z r.
\end{aligned} \tag{22}$$

The components of the trial extra stress function for the external fluid are chosen to be (Wasserman & Slattery 1964)

$$\begin{aligned}
(\tau_{r\theta})_0^* / K\left(\frac{V_\infty}{a}\right)^n &= -A x^B (1-z^2)^{1/2}, \\
(\tau_{rr})_0^* / K\left(\frac{V_\infty}{a}\right)^n &= -(C x^D + C' x^B) z, \\
(\tau_{\theta\theta})_0^* / K\left(\frac{V_\infty}{a}\right)^n &= -(F x^D + F' x^B) z, \\
(\tau_{\phi\phi})_0^* / K\left(\frac{V_\infty}{a}\right)^n &= -(E x^D + E' x^B) z.
\end{aligned} \tag{23}$$

Substituting these trial extra stress functions into the equation of motion for the external fluid, and equating

$$\frac{\partial^2 (p + \rho\phi)_0^*}{\partial r \partial \theta} = \frac{\partial^2 (p + \rho\phi)_0^*}{\partial \theta \partial r}, \tag{24}$$

we get

$$\begin{aligned}
E &= F, \\
E' &= F', \\
A(B-1) &= C' - F',
\end{aligned} \tag{25}$$

and

$$(D-2)(C-F) = 0. \tag{26}$$

Furthermore, since the trace of the extra stress tensor is zero, [23], [25] and [26] require that

$$\begin{aligned}
C &= -2F, \\
C' &= -2F', \\
D &= 2.
\end{aligned} \tag{27}$$

The trial extra stress fields are made to satisfy the jump momentum balance at the interface. If we assume the equilibrium theory of interfacial tension, the sole effect of the interfacial tension is to bring about a discontinuity of the normal stress $(\tau_{rr} - p)$. This manifests itself as a pressure difference $p_i - p_0 = 2\sigma/a$ at each point on the interface. The existence of the interfacial tension, however, does not contribute to the tangential stress $\tau_{r\theta}$ and hence, (Happel & Brenner 1965)

$$(\tau_{r\theta})_i^* = (\tau_{r\theta})_0^* \quad \text{at} \quad r = 1,$$

which yields

$$A = 6C_1X, \tag{28}$$

where the viscosity ratio $X = \eta_i/K(V_\infty/a)^{n-1}$. Using [25] and [27] and substituting the stress functions given by [23] in the equation of motion, the pressure distribution becomes

$$\frac{(p+\rho\phi)_0^*}{K(V_\infty/a)^n} = -z[\{(B-3)A + F'\}x^B + Fx^2]. \tag{29}$$

We can now evaluate H given by [13] using [1], [2], [7], [22], [23] and [29].

Defining the drag coefficient C_d and Reynolds number Re as

$$C_d = \frac{2F_d}{\pi a^2 \rho V_\infty^2}, \qquad Re = \frac{(2a)^n V_\infty^{2-n}\rho}{K}, \tag{30}$$

it can be shown that

$$Y = \frac{C_d Re}{24} \geq \frac{2^{(n-2)}(n+1)}{3}\left\{\frac{4}{3}(F-C) - 16C_1^2X - \frac{n}{(n+1)}\left(\frac{1}{2}\right)^{(1-n)/2n}\int_{-1}^{1}\int_0^1 \Pi_{\tau 0}^{*(n+1)/2n} x^4 \,dx\,dz\right\}, \tag{31}$$

where $\Pi_{\tau 0}^*$ is the dimensionless second invariant of the trial extra stress tensor for the external fluid given by

$$\Pi_{\tau 0}^* = (\tau_i^j \tau_j^i)_0^* / \{K(V_\infty/a)^n\}^2 = x^{2B}\{2A^2(1-z^2) + z^2(C'^2 + 2F'^2)\} + x^{B+2}z^2(2CC' + 4FF') + x^B z^2(C^2 + 2F^2). \tag{32}$$

The maximum of the R.H.S. of inequality [31] obtained by the choice of C_1, C and B gives the "bound-on-bound" on Y.

Solution

A numerical technique is employed to evaluate the maximum of the expression to the right of inequality [31]. A search is made on the three variables C_1, C, and B using the method of Rosenbrock (Rosenbrock & Storey 1966) starting with the converged values for an external fluid of lower pseudoplasticity. The search in three mutually orthogonal directions is continued till the successive values of the right side of inequality [31] do not alter by more than 10^{-5}. The maximum so obtained gives the bound-on-bound Y_{LB}.

DISCUSSION

Figure 1 is a plot of the lower bound obtained in the present investigation ($0.01 \leq X \leq 1000$) and those of Wasserman & Slattery (1964) and Yoshioka & Adachi (1973) for a solid sphere. The upper bound shown in the figure was computed by Mohan (1974) by improving on the functional used by Nakano & Tien (1968) to take into consideration the contribution to the work function

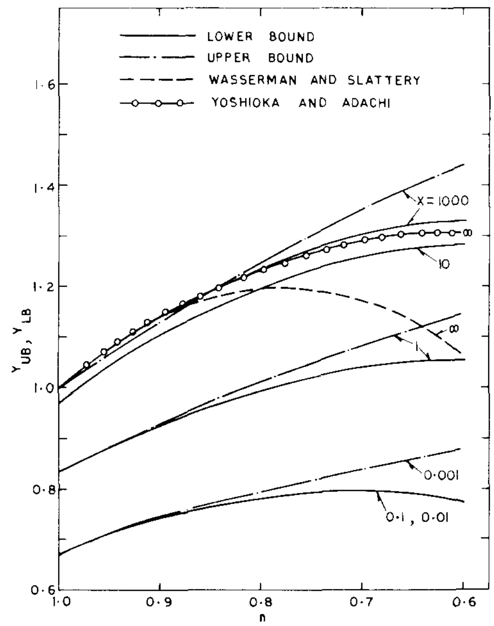

Figure 1. Plot of the upper and lower bounds on Y vs the flow behavior index n for various viscosity ratios X, and the lower bounds due to Wasserman & Slattery (1964) and Yoshioka & Adachi (1973).

from the internal fluid. It is seen that results of Yoshioka & Adachi show good agreement with the results of the present work for a viscosity ratio of 10^3 (solid behavior). However, the results of Wasserman & Slattery are quite low possibly due to the restricted form ($B = 4$) of the trial stress function chosen. The figure also reveals that very close bounds on the drag are obtained, even though the lower bound is a bound-on-bound. Thus, the bound-on-bound approach developed in this work for the lower bound together with that for the upper bound provides a means for developing techniques for analyzing multiphase flow situations involving non-Newtonian fluids.

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